ERROR ESTIMATES OF HIGH-ORDER NUMERICAL METHODS FOR SOLVING TIME FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

Error estimates of some high-order numerical methods for solving time fractional partial differential equations are studied in this paper. We first provide the detailed error estimate of a high-order numerical method proposed recently by Li et al. [21] for solving time fractional partial differential equation. We prove that this method has the convergence order $O(\tau^{3-\alpha})$ for all $\alpha \in (0,1)$ when the first and second derivatives of the solution are vanish at $t = 0$, where τ is the time step size and α is the fractional order in the Caputo sense. We then introduce a new time discretization method for solving time fractional partial differential equations, which has no requirements for the initial values as imposed in Li et al. [21]. We show that this new method also has the convergence order $O(\tau^{3-\alpha})$ for all $\alpha \in (0,1)$. The proofs of the error estimates are based on the energy method developed recently by Lv and Xu [26]. We also consider the space discretization by using the finite element method. Error estimates with convergence order $O(\tau^{3-\alpha} + h^2)$ are proved in the fully discrete case, where h is the space step size. Numerical examples in both one- and two-dimensional cases are given to show that the numerical results are consistent with the theoretical results.

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Key Words and Phrases: time fractional partial differential equations, finite difference method, stability, error estimates

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1. Introduction

Consider the following time fractional partial differential equation

$$ {}_0^C D_t^\alpha u(x,t) - \Delta u(x,t) = f(x,t),\ x \in \Omega,\ t \in [0,T], \tag{1.1} $$

$$ u(x,0) = u_0, \quad x \in \Omega, \tag{1.2} $$

$$ u(x,t) = 0, \quad x \in \partial\Omega,\ t \in [0,T], \tag{1.3} $$

where $0 < \alpha < 1$ and Δ denotes the Laplacian and $\Omega \subset \mathbf{R}^d, d = 1,2,3$ is a bounded and regular domain. Here f is a given function and ${}_0^C D_t^\alpha v(t)$ denotes the Caputo fractional order derivative defined by

$$ {}_0^C D_t^\alpha v(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} v'(s)\, ds, \tag{1.4} $$

where $v'(s) = \frac{dv(s)}{ds}$ denotes the derivative of the first order.

Many engineering and physical problems can be modeled by using (1.1)-(1.3), for example, thermal diffusion in media with fractional geometry [29], highly heterogeneous aquifer [1], underground environmental problems [13], random walks [12, 28], etc.

It is impossible to find the analytic solution of (1.1)-(1.3) in the general case. Therefore we have to construct some numerical methods for solving (1.1)-(1.3), for example, finite difference method, [17], [34], [35], [8], [19], [31], [33], finite element method, [35], [9], [15], [16], [27], [20] and the spectral method, [25], [22], etc. The Caputo fractional derivative (1.4) is a nonlocal operator and the value at time t depends on all the previous values of $v(s), 0 < s < t$. It is natural to introduce some high-order numerical methods for solving (1.1)-(1.3). There are some different approaches to construct high-order schemes to approximate the fractional derivative in the literature, see e.g., [23], [30], [7], [32], [2], [18], [32], [6], [38], [36], [37], [3], [4], [14], [5], [10], [24], etc.

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a time partition of $[0,T]$ and τ the time step size. At $t = t_k, k = 1,2,\ldots,N$, we have

$$ {}_0^C D_t^\alpha v(t_k) = \frac{1}{\Gamma(1-\alpha)} \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} (t_k - s)^{-\alpha} v'(s)\, ds. \tag{1.5} $$

Gao et al. [11] introduced the following way to approximate the Caputo fractional derivative in (1.5): for $k = 1$,

$$ {}_0^C D_t^\alpha v(t_1) \approx \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t_1} (t_1 - s)^{-\alpha} P_1(s)\, ds = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{i=0}^{1} \tilde{w}_{i,1} v(t_{1-i}), $$

and, for $k \geq 2$,

$$
\begin{aligned}
{}_0^C D_t^\alpha v(t_k) &\approx \frac{1}{\Gamma(1-\alpha)} \sum_{j=1}^{k-1} \int_{t_{j-1}}^{t_j} (t_k - s)^{-\alpha} P_2^j(s)\, ds + \int_{t_{k-1}}^{t_k} (t_k - s)^{-\alpha} P_2^{k-1}(s)\, ds \\
&= \frac{\tau^{-\alpha}}{\Gamma(3-\alpha)} \sum_{i=0}^{k} \tilde{w}_{i,k} v(t_{k-i}),
\end{aligned}
$$

where $\tilde{w}_{i,k}$ are some suitable weights and $P_1(s)$ denotes the linear interpolation polynomial defined on the nodes $s = t_0$ and $s = t_1$, $P_2^j(s)$ denote the quadratic interpolation polynomials defined on the nodes $s = t_{j-1}, t_j, t_{j+1}$. The following truncation errors hold

$$
{}_0^C D_t^\alpha v(t_1) = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{i=0}^{1} \tilde{w}_{i,1} v(t_{1-i}) + O(\tau^{2-\alpha}), \tag{1.6}
$$

$$
{}_0^C D_t^\alpha v(t_k) = \frac{\tau^{-\alpha}}{\Gamma(3-\alpha)} \sum_{i=0}^{k} \tilde{w}_{i,k} v(t_{k-i}) + O(\tau^{3-\alpha}),\ k \geq 2. \tag{1.7}
$$

In [11], Gao et al. applied the time discretization schemes (1.6)-(1.7) for solving the time fractional partial differential equation (1.1)-(1.3), but there are no error estimates proved in [11]. Lv and Xu [26] showed that by using the energy method the convergence rate of the time discretization scheme developed in [11] for solving (1.1)-(1.3) is $O(\tau^{3-\alpha})$ for all $\alpha \in (0,1)$.

Recently, Li et al. [21] introduced the following slightly different method from Gao et al. [11] to approximate the Caputo fractional derivative (1.5). On each interval $[t_{j-1}, t_j], j = 1, 2, \ldots, k$, the derivative $v'(s), s \in [t_{j-1}, t_j]$ is approximated by

$$
v'(s) = v'(t_j) + u''(t_j)(s - t_j) + \frac{v'''(t_j)}{2!}(s - t_j)^2 + O\big((s - t_j)^3\big).
$$

Using

$$
v'(t_j) = \frac{u(t_{j+1}) - u(t_{j-1})}{2\tau} - \frac{v'''(t_j)}{3!}\tau^2 + O(\tau^4),
$$

and

$$
v''(t_j) = \frac{v(t_{j+1}) - 2u(t_j) + v(t_{j-1})}{\tau^2} - \frac{v^{(4)}(t_j)}{12}\tau^2 + O(\tau^4),
$$

we get

$$
\begin{aligned}
v'(s) =& \frac{v(t_{j+1}) - v(t_{j-1})}{2\tau} + \frac{v(t_{j+1}) - 2u(t_j) + v(t_{j-1})}{\tau^2}(s - t_j) \\
&- \frac{v'''(t_j)}{3!}\tau^2 + \frac{v'''(t_j)}{2!}(s - t_j)^2 + O\big((s - t_j)^3\big).
\end{aligned} \tag{1.8}
$$

Substituting (1.8) into (1.5), we get

$$
\begin{aligned}
{}_0^C D_t^\alpha v(t_k) =& \frac{\tau^{-\alpha}}{\Gamma(3-\alpha)} \sum_{i=0}^{k-1} w_{1,k-i}\big(v(t_{i+1}) - v(t_{i-1})\big) \\
&+ w_{2,k-i}(v(t_{i+1}) - 2v(t_i) + v(t_{i-1})) + O(\tau^{3-\alpha}),
\end{aligned} \tag{1.9}
$$

where $w_{1,k-i}, w_{2,k-i}$ are defined in (2.16) below. Note that we need to define the values of $v(t_{-1})$ in (1.9). Suppose that we choose $v(t_{-1}) = v(t_0)$, then we get the following approximation scheme

$$
{}_0^C D_t^\alpha v(t_k) = \frac{\tau^{-\alpha}}{\Gamma(3-\alpha)} \sum_{i=0}^{k} \bar{w}_{i,k} v(t_{k-i}) + R_\tau^k, \tag{1.10}
$$

where the weights $\bar{w}_{i,k}, i = 0, 1, \dots, k,\ k = 1, 2, \dots, N$ are defined in (2.13)-(2.15) below. Under the assumptions that $v'(t_0) = v''(t_0) = 0$, we can show that, see [21, (8)],

$$
R_\tau^k = O(\tau^{3-\alpha}),\ k = 1, 2, \dots, N.
$$

Li et al. [21] applied the scheme (1.10) to solve time fractional partial differential equation (1.1)-(1.3) and proved by using the Fourier method that the convergence rate of the numerical method is $O(\tau^{3-\tau})$ for $\alpha \in (0, \alpha_0)$, with $\alpha_0 = 0.37$. In this paper, we shall prove that the convergence rate of the numerical method developed in Li et al. [21] for solving (1.1)-(1.3) is $O(\tau^{3-\tau})$ for all $\alpha \in (0, 1)$. The idea of the proof is based on the energy method developed in [26], see also [23].

We remark that the time discretization scheme (1.10) is easier to implement for solving (1.1)-(1.3) than using the scheme (1.6)-(1.7) since (1.10) is suitable for all $k = 1, 2, \dots, N$, but (1.7) is only suitable for $k \geq 2$. However we have to assume $v'(t_0) = v''(t_0) = 0$ for using (1.10) in order to get the convergence rate $O(\tau^{3-\alpha})$ for all $k = 1, 2, \dots, N$.

Since we use the different approaches to obtain (1.6)-(1.7) and (1.10), respectively, the weights in (1.6)-(1.7) and (1.10) are completely different. Therefore it is natural to consider the error estimates of the numerical methods developed in Li et al. [21] for solving (1.1)-(1.3) by using (1.10) which is one of the objects in the current paper.

Note that the discretization scheme (1.10) must satisfy the initial conditions $v'(t_0) = v''(t_0) = 0$ in order to get the convergence order $O(\tau^{3-\alpha})$. In this paper we will introduce a new scheme which does not require v satisfy $v'(t_0) = v''(t_0) = 0$. We can also show that this scheme has the convergence order $O(\tau^{3-\alpha})$ for all $\alpha \in (0, 1)$. Let us see how to construct this new scheme. For $k = 2, 3, \dots, N$, we will use the following way to approximate the Caputo derivative at t_k,

$$
\begin{aligned}
{}_0^C D_t^\alpha v(t_k) &\approx \int_{t_0}^{t_1} (t_k - s)^{-\alpha} P_2^1(s)\, ds + \frac{1}{\Gamma(1-\alpha)} \sum_{j=2}^{k} \int_{t_{j-1}}^{t_j} (t_k - s)^{-\alpha} P_2^{j-1}(s)\, ds \\
&= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{i=0}^{k} \bar{w}_{i,k} v(t_{k-i}),
\end{aligned}
\tag{1.11}
$$

where $P_2^j(s)$ denotes the quadratic interpolation polynomials defined on the nodes $s = t_{j-1}, t_j, t_{j+1}$ as above. To apply this scheme to solve the time fractional partial differential equation (1.1)-(1.3), we have to obtain the approximate value of the solution at t_1 with the required accuracy by using other numerical methods, then we use this scheme to approximate the solutions at t_k with $k = 2, 3, \ldots, N$. In (1.11), we use the same notations $\bar{w}_{i,k}$ as in (1.10), but they denote the different values. It is easy to show that the truncation error of (1.11) satisfies

$$
{}_0^C D_t^\alpha v(t_k) = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{i=0}^{k} \bar{w}_{i,k} v(t_{k-i}) + O(\tau^{3-\alpha}),\ k = 2, 3, \ldots, N. \tag{1.12}
$$

We remark that the new scheme (1.12) is different from (1.6)-(1.7) and (1.10). In section 3, we apply the scheme (1.12) to solve the time fractional partial differential equation (1.1)-(1.3) and prove that this scheme has the convergence order $O(\tau^{3-\alpha})$ for all $\alpha \in (0, 1)$.

The main contributions of this paper are as follows:

(i). We obtain the properties of the weights in (1.10), i.e., Lemmas **2.1** and **2.2** for the approximation of the Caputo derivatives, which are important to prove the stability and error estimates of the numerical methods for solving (1.1)-(1.3).

(ii). We prove that the convergence rate of the numerical method introduced in Li et al. [21] for solving (1.1)-(1.3) is $O(\tau^{3-\alpha})$ for all $\alpha \in (0, 1)$ by using the energy method in [26].

(iii). We introduce a new time discretization scheme for solving (1.1)-(1.3) by using (1.12) and prove that this scheme has the convergence rate $O(\tau^{3-\alpha})$ for all $\alpha \in (0, 1)$. This scheme does not require that the first and second time derivatives of the solutions are vanish at $t = 0$.

The paper is organized as follows. In Section 1, we consider the time discretization for solving time fractional partial differential equations and prove that the numerical method proposed in [21] has the convergence order $O(\tau^{3-\alpha})$ for all $0 < \alpha < 1$. In Section 2, we consider error estimates of the scheme proposed in [21] for solving time fractional partial differential equations in the fully discrete case where the spatial variables are discretized by using the standard Galerkin finite element method. In Section 3, we consider the error estimate of the new time discretization method constructed by using (1.12) for solving time fractional partial differential equation in the fully discrete case. Finally in Section 4, we give numerical examples to show that the numerical results are consistent with our theoretical results.

By C we denote a positive constant independent of the functions and parameters concerned, but not necessarily the same at different occurrences.

2. Time discretization

In this section, we shall consider the approximation scheme (1.10) introduced in [21] for solving (1.1)-(1.3). It is easy to see that the weights in (1.10) satisfy the following: for $k=1$,

$$\begin{aligned} \bar{w}_{0,1} &= w_{1,1} + w_{2,1}, \\ \bar{w}_{1,1} &= -w_{1,1} - w_{2,1}, \end{aligned} \tag{2.13}$$

and for $k=2$,

$$\begin{aligned} \bar{w}_{0,k} &= w_{1,1} + w_{2,1}, \\ \bar{w}_{1,k} &= w_{1,2} + w_{2,2} - 2w_{2,1}, \\ \bar{w}_{k,k} &= -(w_{1,k} + w_{1,k-1} + w_{2,k} - w_{2,k-1}). \end{aligned} \tag{2.14}$$

and for $k = 3, 4, \dots, N$,

$$\begin{aligned} \bar{w}_{0,k} &= w_{1,1} + w_{2,1}, \\ \bar{w}_{1,k} &= w_{1,2} + w_{2,2} - 2w_{2,1}, \\ \bar{w}_{i,k} &= w_{1,i+1} - w_{1,i-1} + w_{2,i+1} - 2w_{2,i} + w_{2,i-1}, \quad i = 2, 3, \dots, k-1, \\ \bar{w}_{k,k} &= -(w_{1,k} + w_{1,k-1} + w_{2,k} - w_{2,k-1}). \end{aligned} \tag{2.15}$$

Here, with $i = 0, 1, 2, \dots, k-1,\ k = 1, 2, \dots, N$,

$$\begin{aligned} w_{1,k-i} &= \frac{2-\alpha}{2}\Big((k-i)^{1-\alpha} - (k-i-1)^{1-\alpha}\Big), \\ w_{2,k-i} &= (k-i)^{2-\alpha} - (k-i-1)^{2-\alpha} - (2-\alpha)(k-i-1)^{1-\alpha}. \end{aligned} \tag{2.16}$$

Remark **2.1**. We observe that the weights $\bar{w}_{0,k} = \bar{w}_{0,1},\ k = 1, 2, \dots, N$ and $\bar{w}_{1,k} = \bar{w}_{1,2},\ k = 2, 3, \dots, N$. In other words $\bar{w}_{0,k}, k = 1, 2, \dots, N$ take the same value and similarly $\bar{w}_{1,k}, k = 2, 3, \dots, N$ take the same value. For the convenience of the notations, sometimes we denote $\bar{w}_0 := \bar{w}_{0,k}, k = 1, 2, \dots N$ in this work.

By (1.10), the system (1.1)-(1.3) may be written as

$$u(x,t_k)-\kappa_\alpha \Delta u(x,t_k) = \sum_{i=1}^{k} d_{i,k} u(x,t_{k-i}) + \kappa_\alpha f(x,t_k) + \kappa_\alpha R_\tau^k, \quad k=1,2,\dots,N, \tag{2.17}$$

where

$$\kappa_\alpha = \frac{\Gamma(3-\alpha)\tau^\alpha}{\bar{w}_0}, \quad d_{i,k} = -\frac{\bar{w}_{i,k}}{\bar{w}_0}, \; i=1,2,\dots,k, \; k=1,2,\dots,N. \tag{2.18}$$

Below we denote $H = L^2(\Omega)$ with the norm $\|\cdot\|$ and the inner product $(\cdot,\cdot)$.

Let $u^k \approx u(x,t_k)$, $k=1,2,\dots,N$ denote the approximate solution of $u(x,t_k)$. We define the following numerical method for solving (1.1)-(1.3), with $f^k = f(x,t_k), k=1,2,\dots,N$,

$$u^k - \kappa_\alpha \Delta u^k = \sum_{i=1}^{k} d_{i,k} u^{k-i} + \kappa_\alpha f^k, \text{ with } u^0 = u_0. \tag{2.19}$$

In particular, for $k=1$, noting that $d_{1,1} = -\frac{\bar{w}_{1,1}}{\bar{w}_0} = 1$,

$$u^1 - \kappa_\alpha \Delta u^1 = u^0 + \kappa_\alpha f^1. \tag{2.20}$$

Theorem **2.1**. *Let $0<\alpha<1$. Let $u^k, k=0,1,\dots,N$ be the approximate solutions defined by* (2.19). *Assume that $u(\cdot,x) \in C^4[0,T]$ and $\frac{\partial u(x,0)}{\partial t} = \frac{\partial^2 u(x,0)}{\partial t^2} = 0$. Then we have*

$$\|u^k\| + \sqrt{\kappa_\alpha}\|\nabla u^k\| \le 4\sqrt{6}(\|u^0\| + \Gamma(1-\alpha)T^\alpha \max_{1\le s\le k} \|f^s\|), \quad k=1,2,\dots,N.$$

To prove Theorem **2.1**, we need the following lemmas.

Lemma **2.1**. Let $0<\alpha<1$ and let $d_{i,k}, i=1,2,\dots,k,\ k=1,2,\dots,N$ be defined as in (2.18). We have

$$0<d_{1,k}<\frac{4}{3}, \tag{2.21}$$

$$-\frac{1}{3}<d_{2,k}<\frac{1}{3}G(\alpha^*)\approx 0.0378, \tag{2.22}$$

$$d_{i,k}>0, \quad \text{for } i=3,4,\dots,k, \text{ with } k=3,4,\dots,N, \tag{2.23}$$

$$\frac{1}{4}(d_{1,k})^2+d_{2,k}>0, \tag{2.24}$$

$$\sum_{i=1}^{k} d_{i,k}=1. \tag{2.25}$$

Proof of lemma **2.1**. We only consider the case for $k=2,3,\dots,N$. Similarly we can consider the case for $k=1$. For (2.21), we have, with $k=2,3,\dots,N$,

$$d_{1,k}=-\frac{\bar{w}_{1,k}}{\bar{w}_{0,k}}=-\frac{w_{1,2}+w_{2,2}-2w_{2,1}}{w_{1,1}+w_{2,1}}=\frac{2}{4-\alpha}\Big(6-\frac{3}{2}\alpha-\frac{6-\alpha}{2^{\alpha}}\Big).$$

By simple calculation, we may show that $0<d_{1,k}<\frac{4}{3}$ for $0<\alpha<1$.

For (2.22), we have

$$d_{2,k}=\frac{1}{4-\alpha}\big(6(6-\alpha)2^{-\alpha}-3(8-\alpha)3^{-\alpha}-3(4-\alpha)\big).$$

Let $G(\alpha)=6(6-\alpha)2^{-\alpha}-3(8-\alpha)3^{-\alpha}-3(4-\alpha)$ for $0<\alpha<1$. By using the basic calculus, we may show that, with $\alpha^*\approx 0.17221918$,

$$-\frac{1}{3}<d_{2,k}<\frac{1}{4-\alpha}G(\alpha^*)<\frac{1}{3}G(\alpha^*)\approx 0.0378,$$

which is (2.22).

We now consider (2.23). For $i=3,4,\cdots,k-1$ with $k=4,5,\dots,N$, we have

$$d_{i,k}=-\frac{\bar{w}_{i,k}}{\bar{w}_{0,k}}=-\frac{2}{4-\alpha}\big(w_{1,i+1}-w_{1,i-1}+w_{2,i+1}-2w_{2,i}+w_{2,i-1}\big).$$

Here

$$\begin{aligned}
&w_{1,i+1}-w_{1,i-1}+w_{2,i+1}-2w_{2,i}+w_{2,i-1}\\
&=\frac{2-\alpha}{2}\Big((i+1)^{1-\alpha}-2i^{1-\alpha}+(i-1)^{1-\alpha}\Big)+\Big((i+1)^{2-\alpha}-2i^{2-\alpha}+(i-1)^{2-\alpha}\Big)\\
&\quad-\Big[\frac{2-\alpha}{2}\Big(i^{1-\alpha}-2(i-1)^{1-\alpha}+(i-2)^{1-\alpha}\Big)+\Big(i^{2-\alpha}-2(i-1)^{2-\alpha}+(i-2)^{2-\alpha}\Big)\Big]\\
&=g(i)-g(i-1),
\end{aligned}$$

where, with $x \geq 2$,

$$g(x) = \frac{2-\alpha}{2}\Big((x+1)^{1-\alpha} - 2x^{1-\alpha} + (x-1)^{1-\alpha}\Big) + \Big((x+1)^{2-\alpha} - 2x^{2-\alpha} + (x-1)^{2-\alpha}\Big).$$

By some basic calculations, we may show $g'(x) < 0$ for $x \geq 2$, which implies

$$d_{i,k} = -\frac{2}{4-\alpha}\big(g(i) - g(i-1)\big) > 0, \; i = 3, 4, \cdots, k-1, k = 4, 5, \cdots, N.$$

For $d_{k,k}, k = 3, 4, \ldots, N$, we have

$$d_{k,k} = -\frac{\bar{w}_{k,k}}{\bar{w}_{0,k}} = \frac{2}{4-\alpha}\big(w_{1,k} + w_{1,k-1} + w_{2,k} - w_{2,k-1}\big).$$

Here

$$\begin{aligned}
& w_{1,k} + w_{1,k-1} + w_{2,k} - w_{2,k-1} \\
&= \Big[k^{2-\alpha} - (k-1)^{2-\alpha} + \frac{2-\alpha}{2}\Big(k^{1-\alpha} - (k-1)^{1-\alpha}\Big)\Big] \\
&\quad - \Big[(k-1)^{2-\alpha} - (k-2)^{2-\alpha} + \frac{2-\alpha}{2}\big((k-1)^{1-\alpha} - (k-2)^{1-\alpha}\big)\Big] \\
&= p(k) - p(k-1),
\end{aligned}$$

where

$$p(x) = x^{2-\alpha} - (x-1)^{2-\alpha} + \frac{2-\alpha}{2}\big(x^{1-\alpha} - (x-1)^{1-\alpha}\big), \; x \geq 2.$$

It is easy to show that $p'(x) > 0$ for $x \geq 2$, which implies that $d_{k,k} > 0, \; k = 3, 4, \cdots, N$.

For (2.24), we have, with $k = 2, 3, \ldots, N$,

$$\frac{1}{4}(d_{1,k})^2 + d_{2,k} = \frac{1}{4}\Big(-\frac{\bar{w}_{1,k}}{\bar{w}_{0,k}}\Big)^2 + \Big(-\frac{\bar{w}_{2,k}}{\bar{w}_{0,k}}\Big) = \frac{1}{(4-\alpha)^2}F(\alpha),$$

where

$$\begin{aligned}
F(\alpha) &= 3(6-\alpha)(4-\alpha)2^{-\alpha} + (6-\alpha)^2 4^{-\alpha} - 3(8-\alpha)(4-\alpha)3^{-\alpha} - \frac{3}{4}(4-\alpha)^2 \\
&= (72 - 30\alpha + 3\alpha^2)2^{-\alpha} + (36 - 12\alpha + \alpha^2)4^{-\alpha} - (96 - 36\alpha + 3\alpha^2)3^{-\alpha} - \frac{3}{4}(4-\alpha)^2.
\end{aligned}$$

After some tedious but simple calculation, we may show that $F'(\alpha) > 0$ for $0 < \alpha < \alpha_0$ and $F'(\alpha) < 0$ for $\alpha_0 < \alpha < 1$, which implies that α_0 is the only maximum of $F(\alpha)$. Since $F(0) = 0, F(1) = 1$, we have $F(\alpha) > 0$ for $0 < \alpha < 1$. Hence we get

$$\frac{1}{4}(d_{1,k})^2 + d_{2,k} = \frac{1}{(4-\alpha)^2}F(\alpha) > 0, \text{ for } 0 < \alpha < 1.$$

Finally we estimate (2.25). For $k = 1, 2, \dots, N$, we have, by (1.10),

$$ {}_0^C D_t^\alpha u(x, t_k) = \frac{\tau^{-\alpha}}{\Gamma(3-\alpha)} \sum_{i=0}^{k} \bar{w}_{i,k} u(x, t_{k-i}) + O(\tau^{3-\alpha}). \tag{2.26} $$

Choosing $u(x,t) = 1$ in (2.26), we obtain $\sum_{i=0}^{k} \bar{w}_{i,k} = 0$, which implies that

$$ d_{1,k} + d_{2,k} + \cdots + d_{k-1,k} + d_{k,k} = 1. $$

Together these estimates complete the proof of Lemma **2.1**.

□

We now turn to the stability for $u^k, k = 1, 2, \dots, N$. For $k = 1$, we may estimate separately. For $k = 2, 3, \dots, N$, we will introduce the following new variables:

$$ \bar{u}^k = u^k - \eta u^{k-1}, \quad \eta = \frac{1}{2} d_{1,k},\ k = 2, 3, \dots, N. \tag{2.27} $$

By Remark **2.1**, we see that $d_{1,k} = -\frac{\bar{w}_{1,k}}{\bar{w}_{0,k}}, k = 2, 3, \dots, N$ take the same value. Further we have, see, e.g., [23, (44)],

$$ \bar{u}^k - \kappa_\alpha \Delta u^k = \sum_{i=1}^{k-1} \bar{d}_{i,k} \bar{u}^{k-i} + \bar{d}_{k,k} u^0 + \kappa_\alpha f^k, \quad k = 2, 3, \dots, N, \tag{2.28} $$

where, with $k = 2, 3, \dots, N$,

$$ \bar{d}_{1,k} := \eta, \tag{2.29} $$

$$ \bar{d}_{i,k} := \eta^i + \sum_{j=2}^{i} \eta^{i-j} d_{j,k},\ i = 2, 3, \dots, k. \tag{2.30} $$

Lemma **2.2**. For $0 < \alpha < 1$, the coefficients in (2.28) satisfy, with $k = 2, 3, \dots, N$,

$$ 0 < \eta = \frac{d_{1,k}}{2} < \frac{2}{3}, \tag{2.31} $$

$$ \bar{d}_{i,k} > 0,\ i = 1, 2, \dots, k, \tag{2.32} $$

$$ \eta + \sum_{i=2}^{k} \bar{d}_{i,k} \le 1, \tag{2.33} $$

$$ (\bar{d}_{k,k})^{-1} \le c_0 \tau^{-\alpha}, \quad \text{for some constant } c_0. \tag{2.34} $$

Proof of Lemma **2.2**. The estimate (2.31) follows (2.21) directly. For (2.32), it is obvious that, by (2.27), (2.29), (2.24), $\bar{d}_{1,k} = \eta > 0$, and $\bar{d}_{2,k} =$

$\eta^2 + d_{2,k} = \frac{1}{4}(d_{1,k})^2 + d_{2,k} > 0$. Further we have, by (2.23)

$$\bar{d}_{i,k} = \bar{d}_{i-1,k}\eta + d_{i,k} > 0, \quad i = 3, 4, \ldots, k, \text{ for } k = 3, 4, \ldots, N.$$

We next estimate (2.33). Let $S_k = \eta + \sum_{i=2}^{k} \bar{d}_{i,k}$, we have

$$S_k = \eta\frac{1-\eta^k}{1-\eta} + d_{2,k}\frac{1-\eta^{k-1}}{1-\eta} + \cdots + d_{k-2,k}\frac{1-\eta^3}{1-\eta} + d_{k-1,k}\frac{1-\eta^2}{1-\eta} + d_{k,k},$$

which implies that, by (2.24) and (2.25),

$$\begin{aligned}(1-\eta)S_k &= \eta(1-\eta^k) + d_{2,k}(1-\eta^{k-1}) + d_{3,k} + \cdots + d_{k-1,k} + d_{k,k} \\ &\quad - d_{3,k}\eta^{k-2} - \cdots - d_{k-2,k}\eta^3 - d_{k-1,k}\eta^2 - d_{k,k}\eta \\ &\le \eta(1-\eta^k) + d_{2,k}(1-\eta^{k-1}) + d_{3,k} + \cdots + d_{k-1,k} + d_{k,k} \\ &\le (1-\eta) - \eta^{k-1}(\eta^2 + d_{2,k}) < (1-\eta).\end{aligned}$$

Hence (2.33) follows.

Finally we estimate (2.34). It is easy to consider the case for $k = 2$. Here we only consider the case for $k = 3, 4, \ldots, N$, we then have

$$d_{k,k} = -\frac{\bar{w}_{k,k}}{\bar{w}_{0,k}} = \frac{2}{4-\alpha}(w_{1,k} + w_{1,k-1} + w_{2,k} - w_{2,k-1}).$$

We shall show that

$$w_{1,k} + w_{1,k-1} + w_{2,k} - w_{2,k-1} > (2-\alpha)(1-\alpha)k^{-\alpha}. \tag{2.35}$$

Assume (2.35) holds at the moment, we then have

$$d_{k,k} > \frac{2(2-\alpha)(1-\alpha)k^{-\alpha}}{4-\alpha} = \frac{2(2-\alpha)(1-\alpha)t_k^{-\alpha}}{4-\alpha}\tau^\alpha > \frac{2(2-\alpha)(1-\alpha)T^{-\alpha}}{4-\alpha}\tau^\alpha,$$

which implies that, with $c_0 = \frac{(4-\alpha)T^\alpha}{2(2-\alpha)(1-\alpha)}$,

$$d_{k,k}^{-1} < c_0\tau^{-\alpha}, k = 3, 4, \ldots, N. \tag{2.36}$$

Similarly we may show (2.36) holds also for $k = 2$, which we need in the proof of (2.39).

Thus, by (2.21), (2.23), (2.24), with $k = 3, 4, \ldots, N$,

$$\bar{d}_{k,k} = \eta^k + \sum_{i=2}^{k} \eta^{k-i} d_{i,k} = \eta^{k-2}(\eta^2 + d_{2,k}) + d_{3,k}\eta^{k-3} + \cdots + d_{k-1,k}\eta + d_{k,k} > d_{k,k}, \tag{2.37}$$

which implies that $(\bar{d}_{k,k})^{-1} < d_{k,k}^{-1} < c_0\tau^{-\alpha}$ for some constant c_0.

It remains to show (2.35). In fact, we have

$$
\begin{aligned}
I &= w_{1,k} + w_{1,k-1} + w_{2,k} - w_{2,k-1} - (2-\alpha)(1-\alpha)k^{-\alpha} \\
&= (k-1)^{2-\alpha}\Big(1+\frac{1}{k-1}\Big)^{2-\alpha} - 2(k-1)^{2-\alpha} + (k-1)^{2-\alpha}\Big(1-\frac{1}{k-1}\Big)^{2-\alpha} \\
&\quad + \frac{2-\alpha}{2}(k-1)^{1-\alpha}\Big(1+\frac{1}{k-1}\Big)^{1-\alpha} - (2-\alpha)(k-1)^{1-\alpha} \\
&\quad + \frac{2-\alpha}{2}(k-1)^{1-\alpha}\Big(1-\frac{1}{k-1}\Big)^{1-\alpha} - (2-\alpha)(1-\alpha)(k-1)^{-\alpha}\Big(1+\frac{1}{k-1}\Big)^{-\alpha}.
\end{aligned}
$$

In the estimates below, we will frequently use the binomial expansion, with $\beta \in \mathbf{R}$,

$$
(1+x)^{\beta} = 1 + \beta x + \frac{\beta(\beta-1)}{2!} + \dots, \quad \text{for } |x| < 1. \tag{2.38}
$$

By using (2.38), we have

$$
\begin{aligned}
I =& - 2(k-1)^{2-\alpha} - (2-\alpha)(k-1)^{1-\alpha} \\
&+ (k-1)^{2-\alpha}\Big[1 + (2-\alpha)\Big(\frac{1}{k-1}\Big) + \frac{(2-\alpha)(1-\alpha)}{2!}\Big(\frac{1}{k-1}\Big)^2 + \dots\Big] \\
&+ (k-1)^{2-\alpha}\Big[1 + (2-\alpha)\Big(-\frac{1}{k-1}\Big) + \frac{(2-\alpha)(1-\alpha)}{2!}\Big(-\frac{1}{k-1}\Big)^2 + \dots\Big] \\
&+ \frac{2-\alpha}{2}(k-1)^{1-\alpha}\Big[1 + (1-\alpha)\Big(\frac{1}{k-1}\Big) + \frac{(1-\alpha)(-\alpha)}{2!}\Big(\frac{1}{k-1}\Big)^2 + \dots\Big] \\
&+ \frac{2-\alpha}{2}(k-1)^{1-\alpha}\Big[1 + (1-\alpha)\Big(-\frac{1}{k-1}\Big) + \frac{(1-\alpha)(-\alpha)}{2!}\Big(-\frac{1}{k-1}\Big)^2 + \dots\Big] \\
&- (2-\alpha)(1-\alpha)(k-1)^{-\alpha}\Big[1 + (-\alpha)\Big(\frac{1}{k-1}\Big) + \frac{(-\alpha)(-\alpha-1)}{2!}\Big(\frac{1}{k-1}\Big)^2 + \dots\Big].
\end{aligned}
$$

After some tedious but simple calculation, we obtain

$$
\begin{aligned}
I = & \Big(\frac{1}{2!} - \frac{1}{1!}\Big)(2-\alpha)(1-\alpha)(-\alpha)(k-1)^{-\alpha-1} \\
& + \Big(\frac{2}{4!} - \frac{1}{2!}\Big)(2-\alpha)(1-\alpha)(-\alpha)(-\alpha-1)(k-1)^{-\alpha-2} \\
& + \Big(\frac{1}{4!} - \frac{1}{3!}\Big)(2-\alpha)(1-\alpha)(-\alpha)(-\alpha-1)(-\alpha-2)(k-1)^{-\alpha-3} \\
& + \Big(\frac{2}{6!} - \frac{1}{4!}\Big)(2-\alpha)(1-\alpha)(-\alpha)(-\alpha-1)(-\alpha-2)(-\alpha-3)(k-1)^{-\alpha-4} + \cdots \\
& + \Big(\frac{1}{(2m)!} - \frac{1}{(2m-1)!}\Big)(2-\alpha)(1-\alpha)(-\alpha)\cdots(-\alpha-(2m-2))(k-1)^{-\alpha-(2m-1)} \\
& + \Big(\frac{1}{(2m+2)!} - \frac{1}{(2m)!}\Big)(2-\alpha)(1-\alpha)(-\alpha)\cdots(-\alpha-(2m-1))(k-1)^{-\alpha-(2m)} \\
& + \dots .
\end{aligned}
$$

Further we have, with $(k-1)^{-\alpha-2m} = (k-1)^{-\alpha-(2m-1)}(k-1)^{-1}, m = 1, 2, \dots,$

$$
\begin{aligned}
I = & \Big[\Big(\frac{1}{2!} - \frac{1}{1!}\Big) + \Big(\frac{2}{4!} - \frac{1}{2!}\Big)\frac{-\alpha-1}{k-1}\Big](2-\alpha)(1-\alpha)(-\alpha)(k-1)^{-\alpha-1} \\
& + \Big[\Big(\frac{1}{4!} - \frac{1}{3!}\Big) + \Big(\frac{2}{6!} - \frac{1}{4!}\Big)\frac{-\alpha-3}{k-1}\Big](2-\alpha)(1-\alpha)(-\alpha)(-\alpha-1)(-\alpha-2)(k-1)^{-\alpha-3} \\
& + \cdots \\
& + \Big[\Big(\frac{1}{(2m)!} - \frac{1}{(2m-1)!}\Big) + \Big(\frac{1}{(2m+2)!} - \frac{1}{(2m)!}\Big)\frac{-\alpha-(2m-1)}{k-1}\Big] \\
& \qquad \cdot (2-\alpha)(1-\alpha)\cdots(-\alpha-(2m-2))(k-1)^{-\alpha-(2m-1)} + \cdots \\
= & \sum_{m=1}^{\infty}\Big[\frac{1-2m}{(2m)!} + \frac{2-(2m+1)(2m+2)}{(2m+2)!}\frac{-\alpha-(2m-1)}{k-1}\Big] \\
& \qquad \cdot (-1)^{2m-1}(2-\alpha)(1-\alpha)\alpha\cdots(\alpha+2m-2)(k-1)^{-\alpha-(2m-1)} \\
= & \sum_{m=1}^{\infty}\Big[\frac{2m-1}{(2m)!} + \frac{2-(2m+1)(2m+2)}{(2m+2)!}\frac{\alpha+2m-1}{k-1}\Big] \\
& \qquad \cdot (2-\alpha)(1-\alpha)\alpha\cdots(\alpha+2m-2)(k-1)^{-\alpha-(2m-1)}.
\end{aligned}
$$

After some simple calculation, we may get

$$
\begin{aligned}
I = & \sum_{m=1}^{\infty}\frac{(2m-1)(2m+1)(2m+2)(k-1) + \big[2-(2m+1)(2m+2)\big](\alpha+2m-1)}{(2m+2)!(k-1)} \\
& \qquad \cdot (2-\alpha)(1-\alpha)\alpha\cdots(\alpha+2m-2)(k-1)^{-\alpha-(2m-1)}.
\end{aligned}
$$

Hence (2.35) follows from the fact, with $m \geq 1$,

$$(2m-1)(2m+1)(2m+2)(k-1) + [2-(2m+1)(2m+2)](\alpha+2m-1) > 0.$$

Together these estimates complete the proof of Lemma **2.2**. □

Proof of Theorem **2.1**. We first consider the case for $k=1$. Multiplying $2u^1$ in both sides of (2.20) and integrating on Ω, we have,

$$(u^1, 2u^1) - (\kappa_\alpha \Delta u^1, 2u^1) = (u^0, 2u^1) + (\kappa_\alpha f^1, 2u^1).$$

By using the Cauchy-Schwarz inequality and Young inequality, we have

$$2\|u^1\|^2 + 2\kappa_\alpha \|\nabla u^1\|^2 \leq 2\|u^0\|^2 + \frac{1}{2}\|u^1\|^2 + 2\bar{d}_{2,2}\|(\bar{d}_{2,2})^{-1}\kappa_\alpha f^1\|^2 + \frac{1}{2}\|u^1\|^2,$$

which implies that

$$\begin{aligned}
&\|u^1\|^2 + 2\kappa_\alpha \|\nabla u^1\|^2 \leq 2\|u^0\|^2 + 2\bar{d}_{2,2}\|(\bar{d}_{2,2})^{-1}\kappa_\alpha f^1\|^2 \\
&\leq 2\|u^0\|^2 + 2\big\|(\bar{d}_{2,2})^{-1}\kappa_\alpha f^1\big\|^2 \leq 2\|u^0\|^2 + 2(\Gamma(1-\alpha)T^\alpha)^2\big\|f^1\big\|^2,
\end{aligned}$$

where in the last inequality, we use the fact, by (2.18) and (2.36),

$$(\bar{d}_{2,2})^{-1}\kappa_\alpha \leq c_0 \tau^{-\alpha}\kappa_\alpha = \frac{(4-\alpha)T^\alpha}{2(2-\alpha)(1-\alpha)}\tau^{-\alpha}\frac{\Gamma(3-\alpha)\tau^\alpha}{\bar{w}_{0,k}} \leq \Gamma(1-\alpha)T^\alpha. \tag{2.39}$$

Thus we get

$$\|u^1\| + \sqrt{\kappa_\alpha}\|\nabla u^1\| \leq 4\sqrt{6}(\|u^0\| + \Gamma(1-\alpha)T^\alpha\big\|f^1\big\|). \tag{2.40}$$

Here we choose the bound $4\sqrt{6}$ in order to make sure that this bound is the same bound as obtained in (2.42) for $k = 2, 3, \dots, N$ below.

We now turn to the case for $k = 2, 3, \dots N$. Multiplying $2\bar{u}^k$ in both sides of (2.28) and integrating on Ω, we have

$$(\bar{u}^k, 2\bar{u}^k) + 2\kappa_\alpha(\nabla u^k, \nabla \bar{u}^k) = \sum_{i=1}^{k-1}\bar{d}_{i,k}(\bar{u}^{k-i}, 2\bar{u}^k) + \bar{d}_{k,k}(u^0, 2\bar{u}^k) + \big(\kappa_\alpha f^k, 2\bar{u}^k\big),$$

Noting that

$$2(\nabla u^k, \nabla \bar{u}^k) = (\nabla u^k, \nabla u^k) + (\nabla \bar{u}^k, \nabla \bar{u}^k) - \eta^2(\nabla u^{k-1}, \nabla u^{k-1}), \quad k = 2, 3, \cdots, N,$$

we have, with $(\kappa_\alpha f^k, 2\bar{u}^k) = \bar{d}_{k,k}((\bar{d}_{k,k})^{-1}\kappa_\alpha f^k, 2\bar{u}^k)$,

$$
\begin{aligned}
& 2\|\bar{u}^k\|^2 + \kappa_\alpha\|\nabla u^k\|^2 + \kappa_\alpha\|\nabla\bar{u}^k\|^2 - \kappa_\alpha\eta^2\|\nabla u^{k-1}\|^2 \\
\leq & \sum_{i=1}^{k-1}\bar{d}_{i,k}(\|\bar{u}^{k-i}\|^2 + \|\bar{u}^k\|^2) + \bar{d}_{k,k}(2\|u^0\|^2 + \frac{1}{2}\|\bar{u}^k\|^2) + \bar{d}_{k,k}((\bar{d}_{k,k})^{-1}\kappa_\alpha f^k, 2\bar{u}^k) \\
\leq & \sum_{i=1}^{k-1}\bar{d}_{i,k}(\|\bar{u}^{k-i}\|^2 + \|\bar{u}^k\|^2) + \bar{d}_{k,k}(2\|u^0\|^2 + \frac{1}{2}\|\bar{u}^k\|^2) + \bar{d}_{k,k}(2\|(\bar{d}_{k,k})^{-1}\kappa_\alpha f^k\|^2 + \frac{1}{2}\|\bar{u}^k\|^2) \\
\leq & \sum_{i=1}^{k-1}\bar{d}_{i,k}\|\bar{u}^{k-i}\|^2 + 2\bar{d}_{i,k}\|u^0\|^2 + 2\bar{d}_{k,i}\|(\bar{d}_{k,k})^{-1}\kappa_\alpha f^k\|^2 + \sum_{i=1}^{k}\bar{d}_{i,k}\|\bar{u}^k\|^2,
\end{aligned}
$$

Using the inequality (2.33), we have

$$
\begin{aligned}
\|\bar{u}^k\|^2 + \kappa_\alpha\|\nabla u^k\|^2 \leq & \bar{d}_{1,k}\|\bar{u}^{k-1}\|^2 + \kappa_\alpha\eta^2\|\nabla u^{k-1}\|^2 + \bar{d}_{2,k}\|\bar{u}^{k-2}\|^2 + \cdots + \bar{d}_{k-1,k}\|\bar{u}^1\|^2 \\
& + 2\bar{d}_{k,k}\|u^0\|^2 + 2\bar{d}_{k,k}\|(\bar{d}_{k,k})^{-1}\kappa_\alpha f^k\|^2 \\
\leq & \bar{d}_{1,k}(\|\bar{u}^{k-1}\|^2 + \kappa_\alpha\eta\|\nabla u^{k-1}\|^2) + \bar{d}_{2,k}\|\bar{u}^{k-2}\|^2 + \cdots + \bar{d}_{k-1,k}\|\bar{u}^1\|^2 \\
& + 2\bar{d}_{k,k}\|u^0\|^2 + 2\bar{d}_{k,k}\|(\bar{d}_{k,k})^{-1}\kappa_\alpha f^k\|^2 \\
\leq & \bar{d}_{1,k}(\|\bar{u}^{k-1}\|^2 + \kappa_\alpha\|\nabla u^{k-1}\|^2) + \bar{d}_{2,k}\|\bar{u}^{k-2}\|^2 + \cdots + \bar{d}_{k-1,k}\|\bar{u}^1\|^2 \\
& + 2\bar{d}_{k,k}\|u^0\|^2 + 2\bar{d}_{k,k}\|(\bar{d}_{k,k})^{-1}\kappa_\alpha f^k\|^2
\end{aligned}
$$

Denote the norm, with $k = 1, 2, \dots, N$,

$$\|\bar{u}^k\|_1^2 = \|\bar{u}^k\|^2 + \kappa_\alpha\|\nabla u^k\|^2.$$

We then have, with $k = 2, 3, \dots, N$,

$$\|\bar{u}^k\|_1^2 \leq \bar{d}_{1,k}\|\bar{u}^{k-1}\|_1^2 + \bar{d}_{2,k}\|\bar{u}^{k-2}\|_1^2 + \cdots + \bar{d}_{k-1,k}\|\bar{u}^1\|_1^2 + 2\bar{d}_{k,k}\|u^0\|^2 + 2\bar{d}_{k,k}\|(\bar{d}_{k,k})^{-1}\kappa_\alpha f^k\|^2.$$

We will use mathematical induction to prove, with $k = 2, 3, \dots, N$,

$$\|\bar{u}^k\|_1^2 \leq 6\|u^0\|^2 + 6(\Gamma(1-\alpha)T^\alpha)^2 \max_{1\leq s\leq k}\left\|f^s\right\|^2. \qquad (2.41)$$

Let us first consider the case for $k=2$. We have

$$\begin{aligned}
\|\bar{u}^2\|_1^2 &\le \bar{d}_{1,2}\|\bar{u}^1\|_1^2 + 2\bar{d}_{2,2}\|u^0\|^2 + 2\bar{d}_{2,2}\|(\bar{d}_{2,2})^{-1}\kappa_\alpha f^2\|^2 \\
&= \bar{d}_{1,2}(\|\bar{u}^1\|^2 + \kappa_\alpha\|\nabla u^1\|^2) + 2\bar{d}_{2,2}\|u^0\|^2 + 2\bar{d}_{2,2}\|(\bar{d}_{2,2})^{-1}\kappa_\alpha f^2\|^2 \\
&\le \bar{d}_{1,2}(2\|u^1\|^2 + 2\eta^2\|u^0\|^2 + \kappa_\alpha\|\nabla u^1\|^2) + 2\bar{d}_{2,2}\|u^0\|^2 + 2\bar{d}_{2,2}\|(\bar{d}_{2,2})^{-1}\kappa_\alpha f^2\|^2 \\
&\le 2\bar{d}_{1,2}(\|u^1\|^2 + \kappa_\alpha\|\nabla u^1\|^2) + 2(\bar{d}_{1,2}+\bar{d}_{2,2})\|u^0\|^2 + 2\bar{d}_{2,2}\|(\bar{d}_{2,2})^{-1}\kappa_\alpha f^2\|^2 \\
&\le 2(2\|u^0\|^2 + 2\max_{1\le s\le k}\left\|(\bar{d}_{2,2})^{-1}\kappa_\alpha f^s\right\|^2) + 2\|u^0\|^2 + 2\max_{1\le s\le k}\left\|(\bar{d}_{2,2})^{-1}\kappa_\alpha f^s\right\|^2 \\
&\le 6\|u^0\|^2 + 6\max_{1\le s\le k}\left\|(\bar{d}_{2,2})^{-1}\kappa_\alpha f^s\right\|^2 \\
&\le 6\|u^0\|^2 + 6(\Gamma(1-\alpha)T^\alpha)^2\max_{1\le s\le k}\left\|f^s\right\|^2.
\end{aligned}$$

Assume that the following holds for $m=2,3,\cdots,k-1,\ k=3,4,\ldots,N$,

$$\|\bar{u}^m\|_1^2 \le 6\|u^0\|^2 + 6(\Gamma(1-\alpha)T^\alpha)^2\max_{1\le s\le m}\left\|f^s\right\|^2.$$

For $m=k$, we have

$$\begin{aligned}
\|\bar{u}^k\|_1^2 &\le \bar{d}_{1,k}\|\bar{u}^{k-1}\|_1^2 + \bar{d}_{2,k}\|\bar{u}^{k-2}\|_1^2 + \cdots + \bar{d}_{k-1,k}\|\bar{u}^1\|_1^2 + 2\bar{d}_{k,k}\|u^0\|^2 + 2\bar{d}_{k,k}\|(\bar{d}_{k,k})^{-1}\kappa_\alpha f^k\|^2 \\
&\le \sum_{i=1}^{k}\bar{d}_{i,k}(6\|u^0\|^2 + 6(\Gamma(1-\alpha)T^\alpha)^2\max_{1\le s\le k}\|f^s\|^2) \\
&\le 6\|u^0\|^2 + 6(\Gamma(1-\alpha)T^\alpha)^2\max_{1\le s\le k}\|f^s\|^2,
\end{aligned}$$

which implies that

$$\|\bar{u}^k\|_1 \le \sqrt{6}(\|u^0\| + \Gamma(1-\alpha)T^\alpha\max_{1\le s\le k}\|f^s\|),\ k=2,3,\ldots,N.$$

In particular, we have

$$\sqrt{\kappa_\alpha}\|\nabla u^k\| \le \sqrt{6}(\|u^0\| + \Gamma(1-\alpha)T^\alpha\max_{1\le s\le k}\|f^s\|),\ k=2,3,\ldots,N.$$

Further we have,

$$\begin{aligned}
\|u^k\| &= \|\bar{u}^k + \eta u^{k-1}\| \le \|\bar{u}^k\| + \|\eta u^{k-1}\| \le \sqrt{6}(\|u^0\| + \Gamma(1-\alpha)T^\alpha\max_{1\le s\le k}\|f^s\|) + \|\eta u^{k-1}\| \\
&\le \cdots\cdots \\
&\le \frac{1}{1-\eta}\sqrt{6}(\|u^0\| + \Gamma(1-\alpha)T^\alpha\max_{1\le s\le k}\|f^s\|) \\
&\le 3\sqrt{6}(\|u^0\| + \Gamma(1-\alpha)T^\alpha\max_{1\le s\le k}\|f^s\|)
\end{aligned}$$

Therefore we obtain, with $k=2,3,\ldots,N$,

$$\|u^k\| + \sqrt{\kappa_\alpha}\|\nabla u^k\| \le 4\sqrt{6}(\|u^0\| + \Gamma(1-\alpha)T^\alpha\max_{1\le s\le k}\|f^s\|). \tag{2.42}$$

Together this with (2.40) completes the proof of Theorem **2.1**.

□

Next we consider the error estimates. Let $e^k = u(x, t_k) - u^k, k = 0, 1, \ldots, N$, subtracting (2.19) from (2.17), we have the following error equation, with $e^0 = 0$,

$$e^k - \kappa_\alpha \Delta e^k = \sum_{i=1}^{k} d_{i,k} e^{k-i} + \kappa_\alpha R_\tau^k, \quad k = 1, 2, \ldots, N. \tag{2.43}$$

Denote

$$\bar{e}^k = e^k - \eta e^{k-1}, \ \eta = \frac{1}{2} d_{1,k}, \ k = 2, 3, \ldots, N.$$

For $k = 1$, we have, noting that $e^0 = 0$,

$$e^1 - \kappa_\alpha \Delta e^1 = \kappa_\alpha R_\tau^1. \tag{2.44}$$

For $k = 2, 3, \ldots, N$, we have, similar as in (2.28),

$$\bar{e}^k - \kappa_\alpha \Delta e^k = \sum_{i=1}^{k-1} \bar{d}_{i,k} \bar{e}^{k-i} + \kappa_\alpha R_\tau^k, \quad k = 2, 3, \ldots, N, \tag{2.45}$$

where $\bar{d}_{i,k}, i = 1, 2, \ldots, k$ are defined as in (2.28).

Following the proof of Theorem **2.1**, we obtain the following error estimates

THEOREM **2.2**. *Let $0 < \alpha < 1$. Let $u(x, t_k)$ and $u^k, k = 0, 1, \ldots, N$ be the exact and the approximate solutions of* (2.17) *and* (2.19), *respectively. Assume that $u(x, \cdot) \in C^4[0, T]$ and $\frac{\partial u(x,0)}{\partial t} = \frac{\partial^2 u(x,0)}{\partial t^2} = 0$. Then there exists a constant $C = C(\alpha, f, T)$ such that*

$$\|u^k - u(x, t_k)\| + \sqrt{\kappa_\alpha} \|\nabla(u^k - u(x, t_k))\| \le C\tau^{3-\alpha}, \ k = 1, 2, \ldots, N.$$

3. The fully Discretization scheme

In this section, we shall consider the fully discretization scheme for solving (1.1)-(1.3). Here we only consider the error estimates for the homogeneous Dirichlet boundary condition. But the error estimates in Theorem **3.1** is also true for the nonhomogeneous Dirichlet boundary condition.

Let $H^1(\Omega), H^2(\Omega)$ denote the standard Sobolev spaces. Let $S_h \subseteq H_0^1(\Omega)$ denote the standard linear finite element space and h the space step size. Here $H_0^1(\Omega) = \{v : v \in H^1(\Omega), v|_{\partial\Omega} = 0\}$. Let $R_h : H_0^1(\Omega) \to S_h$ denote the Ritz projection defined by, for $\forall \varphi \in H_0^1(\Omega)$,

$$(\nabla R_h \varphi, \nabla \chi) = (\nabla \varphi, \nabla \chi), \quad \forall \chi \in S_h.$$

It is well known that, see e.g., [9],

$$\|R_h\varphi-\varphi\|+h\|\nabla(R_h\varphi-\varphi)\| \le Ch^2\|\varphi\|_{H^2(\Omega)}, \ \ \forall \varphi\in H^2(\Omega)\cap H_0^1(\Omega). \quad (3.46)$$

The finite element approximation of (2.19) reads: find $U_h^k \in S_h$, such that, with $U_h^0 = R_h u_0$,

$$(U_h^k,\chi)+\kappa_\alpha(\nabla U_h^k,\nabla\chi) = \sum_{i=1}^{k} d_{i,k}(U_h^{k-i},\chi)+\kappa_\alpha(f^k,\chi), \ \ \forall\chi\in S_h, \quad k=1,2,\dots,N. \quad (3.47)$$

THEOREM **3.1**. *Let $u(x,t_k)$ and $U_h^k, k=1,2,3,\dots,N$ be the exact and the approximate solutions of (2.17) and (3.47), respectively. Assume that $u(x,\cdot)\in C^4[0,T]$ and $\frac{\partial u(x,0)}{\partial t} = \frac{\partial^2 u(x,0)}{\partial t^2} = 0$ and $u(\cdot,t)\in H^2(\Omega)$. Then there exists a constant $C=C(\alpha,f,T)$ such that*

$$\|U_h^k - u(x,t_k)\| + \kappa_\alpha\|\nabla(U_h^k - u(x,t_k))\| \le C\big(h^2+\tau^{3-\alpha}\big), \ k=1,2,\dots,N.$$

Proof of Theorem **3.1**. Denote, with $k=1,2,\dots,N$,

$$U_h^k - u(x,t_k) = U_h^k - R_h u(x,t_k) + R_h u(x,t_k) - u(x,t_k) = \theta^k + \rho^k.$$

By (3.46), we have $\rho^k \le Ch^2\|u\|_{H^2}$. For $\theta_k, k=1,2,\dots,N$, we have

$$(\theta^k,\chi)+\kappa_\alpha(\nabla\theta^k,\nabla\chi) = \sum_{i=1}^{k} d_{i,k}(\theta^{k-i},\chi) - \kappa_\alpha(\delta_\tau^k,\chi), \ \forall\,\chi\in S_h, \quad (3.48)$$

where $\delta_\tau^k = (I-R_h)\big({}_0^C D_t^\alpha u(x,t_k) - R_\tau^k\big) + R_\tau^k$, where R_τ^k is defined by (1.10).

From the triangle inequality, we have

$$\|\delta_\tau^k\| \le \|(I-R_h){}_0^C D_t^\alpha u(x,t_k)\| + \|(I-R_h)R_\tau^k\| + \|R_\tau^k\|.$$

Note that

$$\|R_\tau^k\| \le C\tau^{3-\alpha}, \quad \|(I-R_h)R_\tau^k\| \le Ch^2\tau^{3-\alpha}, \quad \|(I-R_h){}_0^C D_t^\alpha u(x,t_k)\| \le Ch^2,$$

we have

$$\|\delta_\tau^k\| \le C\big(h^2+h^2\tau^{3-\alpha}+\tau^{3-\alpha}\big) \le C(h^2+\tau^{3-\alpha}).$$

Let $\bar\theta^k = \theta^k - \eta\theta^{k-1}$, $\eta = \frac{d_{1,k}}{2}$, $k=1,2,\dots,N$, following the same argument as in the proof of Theorem **2.1**, we have

$$\|\theta^k\| + \kappa_\alpha\|\nabla\theta^k\| \le C\big(h^2+\tau^{3-\alpha}\big), \ k=1,2,\dots,N.$$

Together these estimates complete the proof of Theorem **3.1**. $\square$

4. A new time discretization scheme

In this section, we shall consider the new approximation scheme (1.12). The weights $\bar{w}_{i,k}, i = 0, 1, \dots, k,\ k = 2, 3, \dots, N$ in (1.12) satisfy the following:

$$\bar{w}_{i,k} = \begin{cases} c_0^{(\alpha)}, & i = 0, \\ c_i^{(\alpha)} - c_{i-1}^{(\alpha)}, & 1 \le i \le k-3, \\ b_{k-1}^{(\alpha)} + c_{k-2}^{(\alpha)} - c_{k-3}^{(\alpha)}, & i = k-2, \\ -3b_{k-1}^{(\alpha)} + c_{k-1}^{(\alpha)} - c_{k-2}^{(\alpha)}, & i = k-1, \\ b_{k-2}^{(\alpha)} + b_{k-1}^{(\alpha)} - a_{k-1}^{(\alpha)}, & i = k, \end{cases}$$

where

$$c_i^{(\alpha)} = \begin{cases} a_0^{(\alpha)} + b_0^{(\alpha)}, & i = 0, \\ a_i^{(\alpha)} + b_i^{(\alpha)} - b_{i-1}^{(\alpha)}, & 1 \le i \le k-2, \\ a_i^{(\alpha)} - b_{i-1}^{(\alpha)}, & i = k-1. \end{cases}$$

and

$$a_i^{(\alpha)} = (i+1)^{1-\alpha} - i^{1-\alpha}, \quad 0 \le i \le k-1,$$

and

$$b_i^{(\alpha)} = [(i+1)^{2-\alpha} - i^{2-\alpha}]/(2-\alpha) - [(i+1)^{1-\alpha} + i^{1-\alpha}]/2, \quad i \ge 0.$$

For $k = 2$, we have

$$\bar{w}_{i,k} = \begin{cases} b_1^{(\alpha)} + c_0^{(\alpha)}, & i = 0, \\ -3b_1^{(\alpha)} + c_1^{(\alpha)} - c_0^{(\alpha)}, & i = 1, \\ b_0^{(\alpha)} + b_1^{(\alpha)} - a_1^{(\alpha)}, & i = 2. \end{cases}$$

For $k = 3$, we have

$$\bar{w}_{i,k} = \begin{cases} c_0^{(\alpha)}, & i = 0, \\ b_2^{(\alpha)} + c_1^{(\alpha)} - c_0^{(\alpha)}, & i = 1, \\ -3b_2^{(\alpha)} + c_2^{(\alpha)} - c_1^{(\alpha)}, & i = 2, \\ b_1^{(\alpha)} + b_2^{(\alpha)} - a_2^{(\alpha)}, & i = 3. \end{cases}$$

For $k = 4$, we have

$$\bar{w}_{i,k} = \begin{cases} c_0^{(\alpha)}, & i = 0, \\ c_1^{(\alpha)} - c_0^{(\alpha)}, & i = 1, \\ b_3^{(\alpha)} + c_2^{(\alpha)} - c_1^{(\alpha)}, & i = 2, \\ -3b_3^{(\alpha)} + c_3^{(\alpha)} - c_2^{(\alpha)}, & i = 3, \\ b_2^{(\alpha)} + b_3^{(\alpha)} - a_3^{(\alpha)}, & i = 4. \end{cases}$$

For $k \geq 5$, we have

$$\bar{w}_{i,k} = \begin{cases} c_0^{(\alpha)}, & i = 0, \\ c_i^{(\alpha)} - c_{i-1}^{(\alpha)}, & 1 \leq i \leq k-3, \\ b_{k-1}^{(\alpha)} + c_{k-2}^{(\alpha)} - c_{k-3}^{(\alpha)}, & i = k-2, \\ -3b_{k-1}^{(\alpha)} + c_{k-1}^{(\alpha)} - c_{k-2}^{(\alpha)}, & i = k-1, \\ b_{k-2}^{(\alpha)} + b_{k-1}^{(\alpha)} - a_{k-1}^{(\alpha)}, & i = k. \end{cases}$$

Let us now turn to the finite difference method for solving (1.1)-(1.3) by using the scheme (1.12). For simplicity, we only consider the case for $k \geq 4$ since the weights $\bar{w}_{i,k}$ has the same expressions for all $k \geq 4$. Similarly we can consider the case for $k = 2, 3$.

At $t_k, k = 4, 5, \dots, N$, the equation (1.1)-(1.3) has the form

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{i=0}^{k} \bar{w}_{i,k} u(x, t_{k-i}) - \Delta u(x, t_k) = f(x, t_k) + R_\tau^k, \tag{4.49}$$

with the truncation error $R_\tau^k = O(\tau^{3-\alpha})$.

Denote $\kappa_\alpha = \frac{\Gamma(2-\alpha)\tau^\alpha}{\bar{w}_{0,k}}$ and

$$d_{i,k} = -\frac{\bar{w}_{i,k}}{\bar{w}_{0,k}}, i = 1, 2, \cdots, k, k = 4, 5, \cdots, N. \tag{4.50}$$

The equation (4.49) can be written as

$$u(x,t_k) - \kappa_\alpha \Delta u(x,t_k) = \sum_{i=1}^{k} d_{i,k} u(x, t_{k-j}) + \kappa_\alpha f(x,t_k) + \kappa_\alpha R_\tau^k, \quad k = 4, 5, \cdots, N. \tag{4.51}$$

Let $u^k \approx u(x, t_k)$, $k = 1, 2, \dots, N$ denote the approximate solution of $u(x, t_k)$. Assume that we obtain the starting approximation u^1, u^2, u^3 with the required accuracy by using other methods. To calculate $u^k, k = 4, 5, \dots, N$, we define the following time discretization scheme for solving (1.1)-(1.3):

$$u^k - \kappa_\alpha \Delta u^k = \sum_{i=1}^{k} d_{i,k} u^{k-i} + \kappa_\alpha f(x, t_k), \quad k = 4, 5, \cdots, N. \tag{4.52}$$

We have

LEMMA **4.1**. Let $0 < \alpha < 1$ and let $d_{i,k}, i = 1, 2, \ldots, k,\ k = 4, 5, \ldots, N$ be defined as in (4.50). We have

$$0 < d_{1,k} < \frac{4}{3}, \tag{4.53}$$

$$-\frac{1}{3} < d_{2,k} < \frac{1}{3}G(\alpha^*) \tag{4.54}$$

$$d_{i,k} > 0, \quad \text{for } i = 3, 4, \ldots, k, \text{ with } k = 3, 4, \ldots, N, \tag{4.55}$$

$$\frac{1}{4}(d_{1,k})^2 + d_{2,k} > 0, \tag{4.56}$$

$$\sum_{i=1}^{k} d_{i,k} = 1. \tag{4.57}$$

P r o o f. The proof is similar as the proof of Lemma **2.1**. □

Let us now turn to the stability proof of $u^k, k = 1, 2, \ldots, N$. As in Section 2, the stability of the starting approximations u^1, u^2, u^3 can be considered separately which can be done easily. For the stability of $u^k, k \geq 4$, since $d_{1,k}$ are the same constant for all $k = 4, 5, \ldots, N$, we introduce some new variables

$$\bar{u}^k = u^k - \eta u^{k-1},\ \eta = \frac{1}{2}d_{1,k},\ k = 4, 5, \ldots, N.$$

We then have

$$\bar{u}^k - \kappa_\alpha \Delta u^k = \sum_{i=1}^{k} \bar{d}_{i,k}\bar{u}^{k-i} + \kappa_\alpha R_\tau^k, \quad k = 4, 5, \ldots, N. \tag{4.58}$$

where

$$\begin{aligned} \bar{d}_{1,k} &:= \eta,\ k = 4, 5, \ldots, N, \\ \bar{d}_{i,k} &:= \eta^i + \sum_{l=2}^{i} \eta^{i-l} d_{l,k},\ i = 2, 3, \ldots, k,\ k = 4, 5, \ldots, N. \end{aligned} \tag{4.59}$$

We may use the same ways to prove the corresponding lemmas and theorems as in Section 3. Therefore we only state the corresponding lemmas and theorems below and leave the proofs for the readers.

Lemma **4.2**. Let $0 < \alpha < 1$ and let $\bar{d}_{i,k}, i = 1, 2, \ldots, k$ with $k = 4, 5, \ldots, N$ be defined by (4.59). Then we have

$$0 < \eta = \frac{1}{2} d_{1,k} < \frac{2}{3}, \tag{4.60}$$

$$\bar{d}_{i,k} > 0, \ i = 1, 2, \ldots, k, \tag{4.61}$$

$$\eta + \sum_{i=2}^{k} \bar{d}_{i,k} \leq 1, \tag{4.62}$$

$$(\bar{d}_{k,k})^{-1} \leq c_0 \tau^{-\alpha}, \quad \text{for some constant } c_0. \tag{4.63}$$

Now we come to the main theorem in this section.

Theorem **4.1**. *Let $0 < \alpha < 1$. Let $u^k, k = 0, 1, \ldots, N$ be the approximate solutions defined by (4.52). Assume that $u(\cdot, x) \in C^3[0, T]$. Assume that the starting approximations u^1, u^2, u^3 satisfy the required stability estimates. Then there exists a constant $C = C(\alpha, T)$ such that*

$$\|u^k\| + \sqrt{\kappa_\alpha} \|\nabla u^k\| \leq C(\|u^0\| + \max_{1 \leq s \leq k} \|f^s\|), \quad k = 1, 2, \ldots, N.$$

For the error estimates of (4.52), we have the following

Theorem **4.2**. *Let $0 < \alpha < 1$. Let $u(x, t_k)$ and $u^k, k = 0, 1, \ldots, N$ be the exact and the approximate solutions of (4.51) and (4.52), respectively. Assume that $u(x, \cdot) \in C^3[0, T]$. Further we assume that the starting errors $e^l = u^l - u(x, t_l), l = 1, 2, 3$ satisfy the required accuracy. Then there exists a constant $C = C(\alpha, f, T)$ such that*

$$\|u^k - u(x, t_k)\| + \sqrt{\kappa_\alpha} \|\nabla (u^k - u(x, t_k)\| \leq C \tau^{3-\alpha}, \ k = 1, 2, \ldots, N.$$

Remark **4.1**. Unlike Theorems **2.1** and **2.2**, there are no conditions $\frac{\partial u(x,0)}{\partial t} = \frac{\partial^2 u(x,0)}{\partial t^2} = 0$ in Theorems **4.1** and **4.2**. We also reduce the regularity requirement for the solution as the function of time variable in our new time discretization scheme (4.52). In Theorems **4.1** and **4.2**, we only require $u(x, \cdot) \in C^3[0, T]$, but in Theorems **2.1** and **2.2**, we require $u(x, \cdot) \in C^4[0, T]$.

Remark **4.2**. We can also consider the error estimates of the fully discrete scheme for solving (4.52) and obtain the similar results as in Theorem **3.1**. We omit it here.

5. Numerical simulations

5.1. Time discretization scheme (2.19) proposed in Li et al. [21]

In this subsection, we will consider two numerical examples to investigate the time convergence rates of the numerical method (2.19) proposed in Li et al. [21]. The first example is for the one-dimensional case and the second example is for the two-dimensional case.

Example **5.1**. [21, Example 5.2] Consider, with $0 < \alpha < 1$,

$$ {}_0^C D_t^{\alpha} u(x,t) - \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), \quad x \in [0,1],\ t > 0, \tag{5.64} $$

$$ u(x,0) = u_0(x), \quad x \in [0,1], \tag{5.65} $$

$$ u(0,t) = u(1,t) = 0, \tag{5.66} $$

where $u(x,0) = 0$, $f(x,t) = \frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)} t^{m-\alpha} \sin(2\pi x) + 4\pi^2 t^m \sin(2\pi x)$. The exact solution is $u(x,t) = t^m \sin(2\pi x)$, where $m = 5$. In this case the exact solution $u(x,\cdot) \in C^4[0,T]$ and $\frac{\partial u(x,0)}{\partial t} = \frac{\partial^2 u(x,0)}{\partial t^2} = 0$.

α	τ	The L_2 error	The convergence rate
0.3	$1/2^3$	1.9413e-04	
	$1/2^4$	3.5274e-05	2.4603
	$1/2^5$	6.0162e-06	2.5517
	$1/2^6$	9.9072e-07	2.6023
	$1/2^7$	1.5940e-07	2.6358
0.5	$1/2^3$	5.5802e-04	
	$1/2^4$	1.1304e-04	2.3035
	$1/2^5$	2.1556e-05	2.3907
	$1/2^6$	3.9801e-06	2.4372
	$1/2^7$	7.1940e-07	2.4679
0.9	$1/2^3$	2.0265e-03	
	$1/2^4$	4.9260e-04	2.0406
	$1/2^5$	1.1327e-04	2.1205
	$1/2^6$	2.5316e-05	2.1717
	$1/2^7$	5.5457e-06	2.1906

Table 1. The time convergence rate with $h = 1/2^6$ at $T = 1$ in Example **5.1**

The main purpose of the numerical example is to check the order of convergence of the numerical method with respect to the time step size τ with the different fractional orders α. For various choices of $\alpha \in (0,1)$, we computed the errors at $T=1$. We use the linear finite element space S_h with space step size $h = 1/2^6$ which is sufficiently small such that the error will be dominated by the time discretization component of the method. We choose the time step size $\tau = 1/2^l, l = 3,4,5,6,7$, i.e, we divided the interval $[0,T]$ into $N = 1/\tau$ small intervals with nodes $0 = t_0 < t_1 < \cdots < t_N = 1$. Then we compute the error $e(t_N) = u(x,t_N) - U_h^N$. By Theorem **2.2**, we have

$$\|e(t_N)\| = \|u(x,t_N) - U_h^N\| \le C\tau^{3-\alpha}, \tag{5.67}$$

To observe the order of convergence we shall compute the error $\|e(t_N)\|$ at $t_N = 1$ with respect to the different values of τ. Denote $\|e_{\tau_l}(t_N)\|$ the error at $t_N = 1$ for the time step size τ_l. Let $\tau_l = 1/2^l$ for a fixed $l = 3,4,5,6,7$. We then have

$$\frac{\|e_{\tau_l}(t_N)\|}{\|e_{\tau_{l+1}}(t_N)\|} \approx \frac{C\tau_l^{3-\alpha}}{C\tau_{l+1}^{3-\alpha}} = 2^{3-\alpha},$$

which implies that the order of convergence satisfies $3-\alpha \approx \log_2\Big(\frac{\|e_{\tau_l}(t_N)\|}{\|e_{\tau_{l+1}}(t_N)\|}\Big)$. In Table 1, we compute the estimated orders of convergence for the different values of α. The numerical results are consistent with the theoretical results in Theorem **2.2**. .

Example **5.2**. [21, Example 5.2 in 2-dimensional case] Consider

$$ {}_0^C D_t^\alpha u(x,t) - \Delta u(x,t) = f(x,t), \quad t \in [0,T],\ x \in \Omega, \tag{5.68}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{5.69}$$

$$u(x,t) = q(x,t), \quad t \in [0,T], x \in \partial\Omega, \tag{5.70}$$

where $\Omega = (0,1)\times(0,1)$. The exact solution is $u(x,t) = t^m \sin(2\pi x_1)\sin(2\pi x_2)$ for some $m > 0$ and

$$f(x,t) = \frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)} t^{m-\alpha}\sin(2\pi x_1)\sin(2\pi x_2) + t^m(8\pi^2)\sin(2\pi x_1)\sin(2\pi x_2).$$

Here $q(x,t) = t^m \sin(2\pi x_1)\sin(2\pi x_2)$ and $u_0(x) = 0$. In Table 2, we choose $m = 5$.

For the various choices of $\alpha \in (0,1)$, we compute the errors at $T = 1$. We use the linear finite element space S_h with the space step size $h = 1/2^6$ which is sufficiently small such that the error will be dominated by the time discretization of the method. We choose the time step size $\tau = 1/2^l, l = 3,4,5,6,7$, i.e, we divide the interval $[0,T]$ into $N = 1/\tau$ subintervals with nodes $0 = t_0 < t_1 < \cdots < t_N = 1$. In Table 2, we compute the orders of convergence for the different values of α. The numerical results are consistent with the theoretical results in Theorem **2.2**. .

α	τ	The L^2 error	The convergence rate
0.3	$1/2^3$	2.6121e-5	
	$1/2^4$	3.9008e-6	2.74
	$1/2^5$	5.6947e-7	2.77
	$1/2^6$	8.2783e-8	2.78
	$1/2^7$	1.2035e-8	2.78
0.5	$1/2^3$	9.1162e-5	
	$1/2^4$	1.6239e-5	2.49
	$1/2^5$	2.8516e-6	2.51
	$1/2^6$	4.9885e-7	2.52
	$1/2^7$	8.6983e-8	2.52
0.9	$1/2^3$	5.8723e-4	
	$1/2^4$	1.4046e-4	2.06
	$1/2^5$	3.3137e-5	2.08
	$1/2^6$	7.7557e-6	2.09
	$1/2^7$	1.7963e-6	2.11

TABLE 2. The time convergence rate with $h = 1/2^6$ at $T = 1$ in Example **5.2**

5.2. The new time discretization scheme (4.52)

In this subsection, we will consider one numerical example to investigate the time convergence rates of the numerical method (4.52) proposed in this paper.

EXAMPLE **5.3**. Consider

$$ {}_0^R D_t^\alpha u(x,t) - \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), \quad t \in [0,T], \ 0 < x < 1, \tag{5.71} $$

$$ u(x,0) = 0, \quad x \in \Omega, \tag{5.72} $$

$$ u(0,t) = u(1,t) = 0, \quad t \in [0,T]. \tag{5.73} $$

The exact solution is $u(x,t) = \sin \pi t \sin \pi x$ which does not satisfy $\frac{\partial u(x,0)}{\partial t} = 0$. We use the same notations as in the experiments in Example **5.1**. We compute the starting value u^1 by using the exact solution which we know in this example. We then use (4.52) to compute other approximate solutions u^k with $k = 2, 3, \ldots, N$. In Table 3, we observe that the numerical results are consistent with the theoretical results.

α	τ	The L_2 error	The time convergence order
0.3	$1/2^3$	3.0459e-04	
	$1/2^4$	5.3478e-05	2.5195
	$1/2^5$	8.9463e-06	2.5820
	$1/2^6$	1.4452e-06	2.6230
	$1/2^7$	2.2540e-07	2.6475
0.5	$1/2^3$	7.7186e-04	
	$1/2^4$	1.4957e-04	2.3549
	$1/2^5$	2.7967e-05	2.4546
	$1/2^6$	5.1138e-06	2.4346
	$1/2^7$	9.1567e-07	2.4864
0.9	$1/2^3$	3.1984e-03	
	$1/2^4$	7.8877e-04	2.0164
	$1/2^5$	1.8945e-04	2.1495
	$1/2^6$	4.4656e-05	2.0845
	$1/2^7$	1.0437e-05	2.1065

TABLE 3. The time convergence rate with $h = 1/2^6$ at $T = 1$ in Example **5.3**

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